Grade 7/8 Math Circles

February 20, 21, 22, 23, 2023

Math in Musical Scales - Solutions

Exercise Solutions

Exercise 1

Determine whether or not the following fractions are in simplest form. If they aren't, write them in simplest form.

a) $\frac{2}{3}$ b) $\frac{3}{9}$ c) $\frac{64}{68}$ d) $\frac{5}{10}$ e) $\frac{36}{63}$ f) $\frac{7}{8}$ g) $\frac{12}{36}$ h) $\frac{57}{91}$

Exercise 1 Solution

a) Simplest form

b) $\frac{3}{9} = \frac{3\times1}{3\times3} = \frac{1}{3}$

c) $\frac{64}{68} = \frac{4\times16}{4\times17} = \frac{16}{17}$

d) $\frac{5}{10} = \frac{5\times1}{5\times2} = \frac{1}{2}$

e) $\frac{36}{63} = \frac{9\times4}{9\times7} = \frac{4}{7}$

f) Simplest form

g) $\frac{12}{36} = \frac{12\times1}{12\times3} = \frac{1}{3}$

h) Simplest form

Exercise 2

Given that the frequency of the tonic is 300 Hz, find the rest of the notes in the Pythagorean scale.

Exercise 2 Solution

- do: 300 Hz

- re: 300 Hz $\times\frac{9}{8} = \frac{300\times9}{8} = \frac{2700}{8} = 337.5$ Hz
- mi: 300 Hz $\times\frac{81}{64} = \frac{300\times81}{64} = \frac{24300}{64} \approx 379.69$ Hz
- fa: 300 Hz $\times\frac{4}{3} = \frac{300\times4}{3} = \frac{1200}{3} = 400$ Hz
- so: 300 Hz $\times\frac{3}{2} = \frac{300\times3}{2} = \frac{900}{2} = 450$ Hz
- la: 300 Hz $\times\frac{27}{16} = \frac{300\times27}{16} = \frac{8100}{16} = 506.25$ Hz
- ti: 300 Hz $\times\frac{243}{128} = \frac{300\times243}{128} = \frac{72900}{128} \approx 569.53$ Hz
- do: 300 Hz $\times 2 = 600$ Hz

Exercise 3

Given that the frequency of the tonic is 280 Hz, find the rest of the notes in the just intonation scale.

Exercise 3 Solution

- do: 280 Hz
- re: 280 Hz $\times\frac{9}{8} = \frac{280\times9}{8} = \frac{2520}{8} = 315$ Hz
- mi: 280 Hz $\times\frac{5}{4} = \frac{280\times5}{4} = \frac{1400}{4} = 350$ Hz
- fa: 280 Hz $\times\frac{4}{3} = \frac{280\times4}{3} = \frac{1120}{3} \approx 373.33$ Hz
- so: 280 Hz $\times\frac{3}{2} = \frac{280\times3}{2} = \frac{840}{2} = 420$ Hz
- la: 280 Hz $\times\frac{5}{3} = \frac{280\times5}{3} = \frac{1400}{3} \approx 466.67$ Hz
- ti: 280 Hz $\times\frac{15}{8} = \frac{280\times15}{8} = \frac{4200}{8} = 525$ Hz
- do: 280 Hz $\times 2 = 560$ Hz

Exercise 4

Find the missing notes in the E scale from Example F.

Exercise 4 Solution

- 372.5 Hz is greater than 348.8 Hz (F) so the second note is F♯
- 419.1 Hz is greater than 392.4 Hz (G) so the third note is G♯
- 558.7 Hz is greater than 523.2 Hz (C_2) so the sixth note is C♯
- 628.6 Hz is greater than $294.3 \times 2 = 588.6$ Hz (D_2) so the seventh note is D♯

The completed scale is E, F♯, G♯, A, B, C♯, D♯, E.

Exercise 5

Find the intervals of the just intonation system.

Exercise 5 Solution

As a reminder, just intonation is as follows:

do	re	mi	fa	so	la	ti	do
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2

- second: $\frac{9}{8} \div 1 = \frac{9}{8}$
- third: $\frac{5}{4} \div \frac{9}{8} = \frac{5}{4} \times \frac{8}{9} = \frac{5\times8}{4\times9} = \frac{40}{36} = \frac{10}{9}$
- fourth: $\frac{4}{3} \div \frac{5}{4} = \frac{4}{3} \times \frac{4}{5} = \frac{4\times4}{3\times5} = \frac{16}{15}$
- fifth: $\frac{3}{2} \div \frac{4}{3} = \frac{3}{2} \times \frac{3}{4} = \frac{3\times3}{2\times4} = \frac{9}{8}$
- sixth: $\frac{5}{3} \div \frac{3}{2} = \frac{5}{3} \times \frac{2}{3} = \frac{5\times2}{3\times3} = \frac{10}{9}$
- seventh: $\frac{15}{8} \div \frac{5}{3} = \frac{15}{8} \times \frac{3}{5} = \frac{15\times3}{8\times5} = \frac{45}{40} = \frac{9}{8}$
- eighth: $2 \div \frac{15}{8} = \frac{2}{1} \times \frac{8}{15} = \frac{2\times8}{1\times15} = \frac{16}{15}$

Thus, the intervals of the just intonation system are:

$$| \frac{9}{8} | \frac{10}{9} | \frac{16}{15} | \frac{9}{8} | \frac{10}{9} | \frac{9}{8} | \frac{16}{15} |$$

Problem Set Solutions

1. Complete the following operations. Reduce all fractions to simplest form. (As a challenge, solve without a calculator).

   a) $\frac{24}{25} \div \frac{1}{2}$
   b) $\frac{7}{10} \times \frac{3}{16}$
   c) $\frac{5}{9} \div \frac{37}{40}$
   d) $\frac{1}{3} \times \frac{9}{7}$
   e) $\frac{6}{13} \div 2$
   f) $\frac{25}{17} \times \frac{12}{5}$

   Solution:

   a) $\frac{24}{25} \div \frac{1}{2} = \frac{24}{25} \times \frac{2}{1} = \frac{24 \times 2}{25} = \frac{48}{25}$
   b) $\frac{7}{10} \times \frac{3}{16} = \frac{7 \times 3}{10 \times 16} = \frac{21}{160}$
   c) $\frac{5}{9} \div \frac{37}{40} = \frac{5}{9} \times \frac{40}{37} = \frac{5 \times 40}{9 \times 37} = \frac{200}{333}$
   d) $\frac{1}{3} \times \frac{9}{7} = \frac{9}{3 \times 7} = \frac{9}{21} = \frac{3}{7}$
   e) $\frac{6}{13} \div 2 = \frac{6}{13} \times \frac{1}{2} = \frac{6}{13 \times 2} = \frac{6}{26} = \frac{3}{13}$
   f) $\frac{25}{17} \times \frac{12}{5} = \frac{25 \times 12}{17 \times 5} = \frac{300}{85} = \frac{60}{17}$

2. Determine whether the following notes are a tone or a semitone apart. Looking at the keys on a piano may help for this question.

   a) C and D
   b) B and C
   c) C♯ and E♭
   d) E♯ and G
   e) G♯ and A♯
   f) B and A♯

   Solution:

   a) Tone

b) Semitone

c) Tone

d) Tone

e) Tone

f) Semitone

3. Find the missing values of the just scale in the table below.

do	**re**	**mi**	**fa**	**so**	**la**	**ti**	**do**
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
150 Hz							300 Hz

Solution:

- re: 150 Hz $\times \frac{9}{8} = \frac{150\times 9}{8} = \frac{1350}{8} = 168.75$ Hz
- mi: 150 Hz $\times \frac{5}{4} = \frac{150\times 5}{4} = \frac{750}{4} = 187.5$ Hz
- fa: 150 Hz$\times \frac{4}{3} = \frac{150\times 4}{3} = \frac{600}{3} = 200$ Hz
- so: 150 Hz$\times \frac{3}{2} = \frac{150\times 3}{2} = \frac{450}{2} = 225$ Hz
- la: 150 Hz$\times \frac{5}{3} = \frac{150\times 5}{3} = \frac{750}{3} = 250$ Hz
- ti: 150 Hz$\times \frac{15}{8} = \frac{150\times 15}{8} = \frac{2250}{8} = 281.25$ Hz

So the completed table should look like this:

do	**re**	**mi**	**fa**	**so**	**la**	**ti**	**do**
1	$\frac{9}{8}$	$\frac{5}{4}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{5}{3}$	$\frac{15}{8}$	2
150 Hz	168.75 Hz	187.5 Hz	200 Hz	225 Hz	250 Hz	281.25 Hz	300 Hz

4. Find the missing values of the Pythagorean scale in the table below.

do	**re**	**mi**	**fa**	**so**	**la**	**ti**	**do**
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
200 Hz							400 Hz

Solution:

- re: 200 Hz $\times \frac{9}{8} = \frac{200\times 9}{8} = \frac{1800}{8} = 225$ Hz
- mi: 200 Hz $\times \frac{81}{64} = \frac{200\times 81}{64} = \frac{16200}{64} \approx 253.13$ Hz
- fa: 200 Hz $\times \frac{4}{3} = \frac{200\times 4}{3} = \frac{800}{3} \approx 266.67$ Hz
- so: 200 Hz $\times \frac{3}{2} = \frac{200\times 3}{2} = \frac{600}{2} = 300$ Hz
- la: 200 Hz $\times \frac{27}{16} = \frac{200\times 27}{16} = \frac{5400}{16} = 337.5$ Hz
- ti: 200 Hz $\times \frac{243}{128} = \frac{200\times 243}{128} = \frac{48600}{128} \approx 379.69$ Hz

So the completed table should look like this:

do	**re**	**mi**	**fa**	**so**	**la**	**ti**	**do**
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
200 Hz	225 Hz	253.13 Hz	266.67 Hz	300 Hz	337.5 Hz	379.69 Hz	400 Hz

5. Transpose the following Pythagorean scale with a tonic of B♭ to have a tonic of F. Find all of the notes for the new scale.

B♭	**C**	**D**	**E♭**	**F**	**G**	**A**	**B♭**
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
58 Hz							116 Hz

Solution: First, find frequencies for B♭ scale:

- C: 58 Hz $\times \frac{9}{8} = \frac{58\times 9}{8} = \frac{522}{8} = 65.25$ Hz
- D: 58 Hz $\times \frac{81}{64} = \frac{58\times 81}{64} = \frac{4698}{64} \approx 73.41$ Hz
- E♭: 58 Hz $\times \frac{4}{3} = \frac{58\times 4}{3} = \frac{232}{3} \approx 77.33$ Hz
- F: 58 Hz $\times \frac{3}{2} = \frac{58\times 3}{2} = \frac{174}{2} = 87$ Hz
- G: 58 Hz $\times \frac{27}{16} = \frac{58\times 27}{16} = \frac{1566}{16} \approx 97.88$ Hz
- A: 58 Hz $\times \frac{243}{128} = \frac{58\times 243}{128} = \frac{14094}{128} \approx 110.11$ Hz

The completed B♭ table looks like this:

B♭	C	D	E♭	F	G	A	B♭
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
58 Hz	65.25 Hz	73.41 Hz	77.33 Hz	87 Hz	97.88 Hz	110.11 Hz	116 Hz

Calculate new notes with tonic of F (87 Hz):

- re: 87 Hz $\times\frac{9}{8} = \frac{87\times9}{8} = \frac{783}{8} \approx 97.88$ Hz
- mi: 87 Hz $\times\frac{81}{64} = \frac{87\times81}{64} = \frac{7047}{64} \approx 110.11$ Hz
- fa: 87 Hz $\times\frac{4}{3} = \frac{87\times4}{3} = \frac{348}{3} = 116$ Hz
- so: 87 Hz $\times\frac{3}{2} = \frac{87\times3}{2} = \frac{261}{2} = 130.5$ Hz
- la: 87 Hz $\times\frac{27}{16} = \frac{87\times27}{16} = \frac{2349}{16} \approx 146.81$ Hz
- ti: 87 Hz $\times\frac{243}{128} = \frac{87\times243}{128} = \frac{21141}{128} \approx 165.16$ Hz

We notice that 97.88 Hz is G, 110.11 Hz is A, 116 Hz is B♭, 130.5 Hz $= 2 \times 65.25$ Hz so 130.5 Hz is C and 146.81 Hz $\approx 2 \times 73.41$ Hz so 146.81 Hz is D. The only note we are missing is the second last one. From the Pythagorean tuning interval pattern, we know the missing note is a tone above D and a semitone below F. Looking at a piano, we realize that this missing note is E. So the completed F scale looks like this:

F	G	A	B♭	C	D	E	F
1	$\frac{9}{8}$	$\frac{81}{64}$	$\frac{4}{3}$	$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$	2
87 Hz	97.88 Hz	110.11 Hz	116 Hz	130.5 Hz	146.81 Hz	165.16 Hz	116 Hz

6. Determine whether the following scale is in the Pythagorean tuning system or the just intonation system. (Hint: look at the intervals).

do	re	mi	fa	so	la	ti	do
50 Hz	56.25 Hz	62.5 Hz	$66\frac{2}{3}$ Hz	75 Hz	$83\frac{1}{3}$ Hz	93.75 Hz	100 Hz

Solution: The just intonation intervals are as follows:

$$| \frac{9}{8} | \frac{10}{9} | \frac{16}{15} | \frac{9}{8} | \frac{10}{9} | \frac{9}{8} | \frac{16}{15} |$$

The Pythagorean tuning intervals are as follows:

$$| \frac{9}{8} | \frac{9}{8} | \frac{256}{243} | \frac{9}{8} | \frac{9}{8} | \frac{9}{8} | \frac{256}{243} |$$

Since the seventh is different between the two tuning systems, let's calculate it from the given scale.
$\frac{100}{93.75} = \frac{10000}{9375} = \frac{625\times16}{625\times15} = \frac{16}{15}$
So the given scale is in the just intonation system.

7. Describe the benefits and drawbacks of Pythagorean tuning and just intonation.

Solution: Benefits of Pythagorean tuning:

- Playable in any key
- Each key has a unique sound
- Has a semitone-tone pattern

Drawbacks of Pythagorea tuning:

- Large numerators and denominators
- Produces wolf intervals in certain keys
- Cannot be repetedly transposed

Benefits of just intonation:

- Has a pure and pleasant sound in some keys
- Each key has a unique sound
- Small numerators and denominators for calculations

Drawbacks of just intonation:

- Sounds "wrong" in some keys
- Does not have a semitone-tone pattern
- Certain keys are unplayable on common instruments

8. If you wanted to transpose a song over and over again, which tuning system would you choose? Why?

Solution: I would choose equal temperament. Pythagorean tuning creates wolf intervals in some keys and just intonation doesn't work for certain keys. I know that equal temperament can be played in all keys so we are able to transpose to any key.